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SOME UNIVERSAL SETS OF TERMS

BY

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ABSTRACT. For every Π_2^1 class of cardinals containing 0 and 1, there exists a finite set T of terms, such that X is precisely the class of cardinals in which T is universal.

Around 1935, W. Sierpiński discovered [42] the remarkable fact that if $(f_1, f_2, \ldots)$ is any sequence of functions on an infinite set E (i.e. f_i: $E \to E$ for each i), then there exist two functions g_1, g_2: $E \to E$ such that each f_i is in the semigroup generated by g_1 and g_2 under composition of functions. In fact, according to a simplification found soon afterward by Banach [3], we may always take g_1 and g_2 so that the f_i are given by these formulas:

$$f_1(x) = g_1 g_2 g_1 g_2(x), \quad f_2(x) = g_1 g_2^2 g_1 g_2(x), \quad f_3(x) = g_1 g_2^3 g_1 g_2(x), \ldots .$$

For functions of two variables, J. Łoś [27] found a similar result around 1950; in fact he found an infinite sequence, of terms built from a single $g_1(x, y)$, which was capable of representing any sequence of operations on an infinite set. For some more historical comments, some easy proofs, and a good elementary introduction to these ideas, we refer the reader to [34].

Later Mycielski coined the term *universal* for (finite or infinite) sets of terms which, like the above $\{ g_1 g_2^i g_1 g_2 : i \in \omega\}$, can represent any sequence of functions through proper choice of the g_i (a precise definition is in §0 below). All known methods for constructing universal sets of terms relied on only the most basic properties of infinite sets (essentially $|E|^2 = |E|$), and so all known examples were universal on all infinite sets E, i.e. regardless of the value of the infinite cardinality $|E|$. Mycielski suggested refining the notion to *κ-universality* (the above notion restricted to sets E with $|E| = \kappa$), and then he asked whether κ-universality and λ-universality are equivalent for all infinite κ and λ. (This question was reported in Isbell [15], and it really involved, at that time, a single term built from unary functions only–a special case of the problem which remains open today. The general form of the problem, which we are answering here, was stated some ten years later by G. McNulty on p. 205 of [31].) Here, in our main result, we answer "no" to the general question:

THEOREM. *For each Π_2^1 class X of cardinals with $0, 1 \in X$, there exists a finite set T of terms such that*

$$X = \{\kappa : T \text{ is universal in power } \kappa\}.$$

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Moreover, there exists a recursive function f such that we may take $T = f$(any Π^1_2 formula defining X).

For the precise definition of "Π^1_2 class", see §0; for this introduction it suffices to say that the class of all Π^1_2 classes is very rich and contains almost every easily conceived cardinal class, including, for instance: $[0, \aleph_\alpha)$, $\{0, 1\} \cup [\beth_\alpha, \infty)$, $\{0, 1\} \cup \{\kappa: \kappa$ is inaccessible$\}$, $[0, 2^\mu)$, for α any recursive ordinal and μ the first measurable cardinal (but not $\{2^{\aleph_0}\}$ and not $[0, \mu)$–see **[22]** and **[14]**). For a full treatment of Π^1_2 classes see S. J. Garland **[14]**. It is immediate from the definitions (§0) that universality of T in power κ defines a Π^1_2 class containing 0 and 1. Hence our answer to the Mycielski-McNulty question is "best possible" to the extent that one regards "Π^1_2" as well understood.

Given an object E in any category with finite products, we can define the *clone* $C(E)$ (in Lawvere's terminology, an *algebraic theory*) to be the full subcategory spanned by E and all its finite powers E^n. More prosaically, $C(E)$ is the family of all maps $E^n \to E$ (any n) together with some obvious composition operators. Thus $C(E)$ extends the endomorphism monoid End(E) and the automorphism group Aut(E). (And, like monoids and groups, the class of all clones can be described axiomatically; moreover every clone C occurs, within isomorphism, in the category of all algebras with $|C|$ unary operations **[37]**.)

One simple place to form clones is in the category of sets. Here, our result says that *if κ and λ are cardinals which are not Π^1_2-equivalent, then $C(\kappa)$ and $C(\lambda)$ are not elementarily equivalent*; in fact there is a special form of AE-sentence true in one but not in the other. See McKenzie **[28]**, Shelah **[38]** and Pinus **[36]** for some results on the elementary equivalence classes of End(κ) and especially Aut(κ). In contrast to the clone situation, there do exist Π^1_2-inequivalent cardinals κ and λ with Aut$(\kappa) \equiv$ Aut(λ). Our results may thus be viewed as contributing to the recently developing study of the first order properties of clones, described in Baldwin and Berman **[2]**. (See also **[53]**.)

The definition of universality of terms makes sense of course in any clone; one case of interest to many mathematicians has been the clone of all continuous real-valued functions of finitely many real variables, i.e. $C(R)$ in the category of continuous maps. Hilbert's thirteenth problem stimulated research which led to the discovery of various terms which are universal for this clone, including formulas to express every continuous function of three variables as a combination of continuous binary functions. For good surveys of this work of Kolmogorov, Arnol'd et al., and also the analytic clone, etc., see Lorentz **[26]**, the introduction to Vitushkin **[52]**, and the published solution for **[34]**. For clones $C(X)$ with X various other topological spaces, see Taylor **[49]** and references given there.

Π^1_2 classes often arise very naturally in mathematical (particularly in algebraic) research, e.g. the class of cardinalities in which a given theory has no simple algebra, or in which it has no Jónsson algebra, etc. In §2 we will review some of these, and in some special cases we will be able to bypass the proof of the main theorem to directly write down a set T of terms which is universal in the appropriate powers. (See 2.7–2.10.) We advise the reader that this section is *easier* than §1, and yet these special examples really contain all the ideas of §1.

The recursiveness assertion of the Theorem has the following Corollary, answering a question which was stated on p. 29 of **[48]** and really goes back to Isbell **[15]**. The proof is immediate from the Theorem and any of various known results about decidability, e.g. Perkins' result [35] that it is undecidable whether a finite set of identities has a model of power $\aleph_0$. (Also see 2.9 below.)

COROLLARY. *It is undecidable whether a finite set T of terms is universal in power $\aleph_0$. Likewise, whether T is universal in all infinite powers.*

It is open whether the Theorem holds for terms built completely from unary operations, i.e. whether End(κ) carries as much information as the clone $C(\kappa)$ as far as these special AE-sentences are concerned. (But cf. Theorem 4 of Pinus **[36]** which tells us that End(κ) $\not\equiv$ End(λ) for κ, λ second-order distinguishable cardinals.) In fact, it is even open whether a finite unary T can distinguish any two infinite cardinals! The question of a unary version of the Corollary also remains open, although it has stimulated important work, and many special cases are now solved. (See the papers of Ehrenfeucht, Silberger and Valente.)

Our proofs seem to make essential use of n-ary operations in T for arbitrarily large finite n. But notice that (in the infinite range) both *the Theorem and its Corollary hold with T restricted to terms involving a single binary operation only*. We need only replace each n-ary operation by a suitable combination of one binary operation (in fact we may use any set of terms which is universal in all infinite powers, e.g. the above-mentioned terms of Łoś). Obviously there is no such easy reduction to the unary case.

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0. Definitions. Let φ be a first order sentence with relation symbols included in $R_1, R_2, \ldots, S_1, S_2, \ldots$ and operations symbols included in $F_1, F_2, \ldots, G_1, G_2, \ldots$. Obviously the truth of the second order sentence

$$\psi = (\forall R_1 R_2 \cdots F_1 F_2 \cdots)(\exists S_1 S_2 \cdots G_1 G_2 \cdots)\varphi$$

in a structure $(A; \ldots)$ depends only on the domain cardinality $\kappa = |A|$. Thus we simply say "κ obeys ψ", in symbols $\kappa \models \psi$. (For this notion, the operation symbols F_i, G_i are clearly dispensable, but we need them below.) Now we can state that the *Π^1_2 class specified by ψ is* $\{\kappa : \kappa \models \psi\}$, and we *define X to be* Π^1_2 iff for some ψ X is the Π^1_2 class specified by ψ.

A set—more properly, a sequence—of terms, $T = (\tau_1, \tau_2, \ldots)$ is *universal in* power κ if it can represent every sequence $(F_1, F_2, \ldots)$ of operations (of the appropriate arities) on a set of cardinality κ. More precisely, let us assume that the operation symbols appearing in T are G_1 (m_1-ary), G_2 (m_2-ary), $\ldots$, and that the variables occurring in τ_i are $x_1, x_2, \ldots, x_{n_i}$. (This last assumption simplifies the definition, represents no loss of generality, and will be true of the terms constructed in the proof of Theorem.) Taking F_i to be a new n_i-ary operation symbol for each i, we say that T is *universal in power* κ iff κ obeys the sentence

$$(\forall F_1 F_2 \cdots)(\exists G_1 G_2 \cdots)\forall x_1 x_2 \cdots \bigwedge_i \tau_i = F_i(x_1, x_2, \ldots).$$

(Note that this definition makes sense for infinite T as well as finite.) Obviously for $\kappa > 1$, any repetition in the sequence T will kill universality, and rearrangements do not affect universality; thus we allow ourselves to speak of universality for T a *set* of terms.

1. Proof of the Theorem. We are given a Π^1_2 class X which is to be represented by universality of terms. Let us suppose that X is defined as in §0 by the second order sentence

$$\psi = (\forall \cdots F \cdots R \cdots)(\exists \cdots G \cdots S \cdots)\varphi$$

with φ first order.

LEMMA 1. *We may assume* φ *is universal.*

PROOF. By Skolemization. Assuming φ prenex, w.l.o.g. of the form $\forall x_1 \exists y_1 \forall x_2 \exists y_2 \ldots$, we replace all occurrences of each y_i by a new $G_i(x_1, \ldots, x_i)$, and then add the G_i's to the list of existentially quantified operation symbols. As is well known, the satisfiability of φ is not altered by this modification. □

LEMMA 2. *We may assume that* ψ *has the form*

$$(\forall \cdots F \cdots)(\exists \cdots G \cdots)(\varphi_1 \to \varphi_2)$$

with φ_1 *and* φ_2 *both positive universal, and with no* G*'s appearing in* φ_1.

(N.b. We expressly mean that *no* relation predicates except equality appear in ψ.)

PROOF. Corresponding to the relations R and S occuring in ψ we will take new operations F_R and G_S (of the same arity), plus new nullary operations F_0, F_1 and one new ternary operation G. We take φ universal, as given by Lemma 1, and moreover we make sure that $\neg$ appears in φ only in forming the negations of atomic formulas. (In particular, $\to$ and $\perp$ do not appear.) We now form φ^* from φ by making the following alterations:

$R(\tau_1, \ldots, \tau_n)$	becomes $F_R(\tau_1, \ldots, \tau_n) = F_1$;
$\neg R(\tau_1, \ldots, \tau_n)$	becomes $F_R(\tau_1, \ldots, \tau_n) = F_0$;
$S(\tau_1, \ldots, \tau_n)$	becomes $G_S(\tau_1, \ldots, \tau_n) = F_1$;
$\neg S(\tau_1, \ldots, \tau_n)$	becomes $G_S(\tau_1, \ldots, \tau_n) = F_0$;
$\neg \sigma = \tau$	becomes $G(\sigma, \sigma, \tau) = F_0 \wedge G(\sigma, \tau, \tau) = F_1$.

Thus φ^* is positive and universal. Now we take φ_1 to be the universal closure of

$$\bigwedge_R \left[F_R(x_1, x_2, \ldots) = F_0 \vee F_R(x_1, x_2, \ldots) = F_1 \right],$$

and we take φ_2 to be the universal closure of

$$(F_0 = F_1) \vee \left[\varphi^* \wedge \bigwedge_S (G_S(x_1, x_2, \ldots) = F_0 \vee G_S(x_1, x_2, \ldots) = F_1) \right].$$

Obviously φ_1 and φ_2 have the required syntactic features. It remains to be seen that $\kappa \models \psi \leftrightarrow \kappa \models \psi'$ for all cardinals κ. We are given that 0 and 1 obey ψ, and it is obvious that 0 and 1 also obey $\forall F_0F_1 \cdots \exists \cdots G \cdots (F_0 = F_1)$, which implies ψ'. Thus we may assume $\kappa > 1$. First let us assume that $\kappa \models \psi$. Thus we are given functions $\cdots F \cdots$ (as in ψ), $\cdots F_R \cdots$ (with R as in ψ), F_0 and F_1. If $F_0 = F_1$ or if φ_1 fails, we are done, since then ψ' holds regardless of our choice of G's. Thus we will assume that $F_0 \neq F_1$ and that φ_1 holds. Now for each symbol R of ψ, define a relation

$$R = \{\vec{x}\colon F_R(\vec{x}) = F_1\}.$$

Since ψ is true on κ, there exist operations $\cdots G \cdots$ and relations $\cdots S \cdots$ on κ so that φ holds. Define operations G_S and G on κ *via*

$$G_S(\vec{x}) = \begin{cases} F_0 & \text{if } \vec{x} \notin S, \\ F_1 & \text{if } \vec{x} \in S, \end{cases}$$

$$G(x, y, z) = \begin{cases} F_0 & \text{if } x = y \neq z, \\ F_1 & \text{if } x \neq y = z, \\ \text{anything in the remaining cases.} \end{cases}$$

It should be clear that $(\kappa; \cdots F \cdots, \cdots F_R \cdots, \cdots G \cdots, \cdots G_S \cdots, G) \models \varphi^*$, and so we are done.

Conversely, suppose that $\kappa \models \psi'$. Let us prove that $\kappa \models \psi$. Given operations $\cdots F \cdots$ and operations $\cdots R \cdots$, define operations F_0, F_1 and $\cdots F_R \cdots$ by first picking $F_0 \neq F_1$ (and otherwise arbitrary) and then defining

$$F_R(\vec{x}) = \begin{cases} F_0 & \text{if } \vec{x} \notin R, \\ F_1 & \text{if } \vec{x} \in R. \end{cases}$$

Since ψ' is true on κ, there exist operations $\cdots G, \cdots, \cdots G_S \cdots$ and G on κ so that $\varphi_1 \to \varphi_2$ holds. Now clearly we already have φ_1 true, thus φ_2 holds. Since $F_0 \neq F_1$ we know that φ^* holds and that

$$(*) \qquad \forall x_1x_2 \cdots [G_S(x_1, x_2, \ldots) = F_0 \vee G_S(x_1, x_2, \ldots) = F_1]$$

for each relation symbol S. If we define relations $\cdots S \cdots$ *via* $\vec{x} \in S \leftrightarrow G_S(\vec{x}) = F_1$, then it is easily checked that φ is an easy consequence of φ^* (using $(*)$). □

PROOF OF THE THEOREM RESUMED. By Lemma 2, our Π^1_2 class X is defined by a second order sentence

$$\psi = (\forall \cdots F \cdots)(\exists \cdots G \cdots)(\varphi_1 \to \varphi_2),$$

with φ_1 given by

$$\forall a_1 \cdots a_M \bigvee_{i=1}^{N} \bigwedge_{k=1}^{M} \alpha_{ik} = \beta_{ik}$$

and φ_2 given by

$$\forall x_1 \cdots x_N \bigwedge_{j=1}^{N} \bigvee_{k=1}^{N} \gamma_{jk} = \delta_{jk}.$$

Here we have deliberately chosen disjunctive normal form in one case, and conjunctive in the other, and we have taken N as a single upper index large enough to cover all contingencies, and M is even larger (and will be increased below). The

γ_{jk} and δ_{jk} are terms in $x_1, x_2, \ldots$, while the α_{ik} and β_{ik} are terms in $a_1, a_2, \ldots$. This distinction will presently be important, when the a_i will appear as (unknown) constant functions of T and the x_i will appear as variables of T. Moreover, we know that none of the operations $\cdots G \cdots$ appear in any of the equations $\alpha_{ik} = \beta_{ik}$.

We now claim that we may assume that among the $\cdots F \cdots$ we have F_0, F_1 (nullary), F_2 (quaternary) and F_3 ($2N$-ary) with these special properties:

(a) F_0-F_3 do not appear in φ_2.

(b) φ_1 entails these formulas:

$$F_2(x, x, y, z) = y, \quad F_2(F_0, F_1, y, z) = z,$$
$$x_i = x_{N+i} \rightarrow F_3(x_1, \ldots, x_{2N}) = F_0 \qquad (1 \leqslant i \leqslant N).$$

(c) Any model of φ_1 remains a model of φ_1 upon altering F_0-F_3, so long as formulas (b) continue to hold.

Of course the claim is easily established by simply adding new operations and laws to φ_1 (thereby increasing M). To see that $\{\kappa: \psi \vDash \kappa\}$ is unaltered, it is enough to observe that formulas (b) can be modeled on every set.

Abbreviation: α_i will stand for $(\alpha_{i1}, \ldots, \alpha_{iM})$; similarly β_i, γ_j and δ_j.

DEFINITION. T will consist of these terms:

$$\tau_F = F(x_1, x_2, \ldots) \quad (\text{each } F \text{ of } \psi),$$
$$\tau_{ij} = K_{ij}(x_1, \ldots, x_N, \alpha_i, F_3(\gamma_j, \delta_j)),$$
$$\tau'_{ij} = K_{ij}(x_1, \ldots, x_N, \beta_i, F_1),$$

with $1 \leqslant i, j \leqslant N$. The variables of T are expressly taken to be $x_1, \ldots, x_N$, and the "unknown" function symbols of T are $\cdots F \cdots$ (explicitly appearing), $\cdots G \cdots$ (occurring in γ_{jk} and δ_{jk}), $(N + M + 1)$-ary K_{ij} (new), and the nullary symbols $a_1, \ldots, a_M$ (occurring in α_{ik} and β_{ik}).

The recursiveness assertion of the Theorem is immediate, and so we now investigate the class of cardinals $\kappa \geqslant 2$ for which T is universal in power κ. First let us assume that $\kappa \vDash \psi$ and prove that T is universal in power κ.

Thus we are given operations $\cdots F \cdots$, F_{ij} and F'_{ij} on κ which we must simultaneously represent with the terms $\cdots \tau_F \cdots$, τ_{ij} and τ'_{ij}. This obviously amounts to saying that we are *given* operations $\cdots F \cdots$ and must find operations $\cdots G \cdots$, a_i, and K_{ij} so as to represent F_{ij} by τ_{ij} and F'_{ij} by τ'_{ij}. Looking at the form of τ_{ij} and τ'_{ij}, we see that each of the equations $\tau_{ij} = F_{ij}$ and $\tau'_{ij} = F'_{ij}$ is simply a definition of the operation K_{ij} on a subset of its domain. Specifically, K_{ij} is defined on these two subsets of its domain:

$$A_{ij} = \{(x, \alpha_i, F_3(\gamma_j, \delta_j)): x \in \kappa^N\}, \quad B_{ij} = \{(x, \beta_i, F_1): x \in \kappa^N\}.$$

To complete the proof of κ-universality, we need to see that $a_1, \ldots, a_M$ and operations $\cdots G \cdots$ can be chosen so that, for all i and j, $A_{ij} \cap B_{ij} = \varnothing$.

Case 1. $(\kappa; \cdots F \cdots) \vDash \varphi_1$. In particular, the formulas (b) hold. For $\kappa \geqslant 2$, the first two identities imply $F_0 \neq F_1$. Moreover, the truth of ψ tells us that we can find operations $\cdots G \cdots$ so that $(\kappa; \cdots F \cdots, \cdots G \cdots) \vDash \varphi_2$. Thus for each

j and each $x \in \kappa^N$, we have $\gamma_{j1} = \delta_{j1}$ or $\gamma_{j2} = \delta_{j2}$ or $\cdots$ or $\gamma_{jN} = \delta_{jN}$. Hence by (b) we have $F_3(\gamma_j, \delta_j) = F_0$, and thus for each i and j,

$$A_{ij} = \{(x, \alpha_i, F_0): x \in \kappa^N\}$$

which is disjoint from B_{ij} (since $F_0 \neq F_1$), regardless of our choice of $a_1, \ldots, a_M$.

Case 2. $(\kappa; \cdots F \cdots) \vDash \neg\varphi_1$. Then we may choose $a_1, \ldots, a_M$ so that, for each i, $\alpha_i \neq \beta_i$. Clearly this makes A_{ij} disjoint from B_{ij} for all i and j, regardless of our choice of the operations $\cdots G \cdots$. This completes our proof that T is universal in power κ.

Conversely, let us suppose that T is universal in power $\kappa \geqslant 2$ and prove that $\kappa \vDash \psi$. Thus we are given operations $\cdots F \cdots$ on κ obeying φ_1, and we wish to find operations $\cdots G \cdots$ so that φ_2 holds. By (a) and (c), we may change F_0-F_3 to the following functions:

F_0, F_1 any two distinct constants;

$$F_2(x, y, z, w) = \begin{cases} z & \text{if } x = y, \\ w & \text{if } x \neq y; \end{cases}$$

$$F_3(x_1, \ldots, x_{2N}) = \begin{cases} F_0 & \text{if } x_1 = x_{N+1} \text{ or } x_2 = x_{N+2} \text{ or } \cdots \text{ or } x_N = x_{2N}, \\ F_1 & \text{otherwise.} \end{cases}$$

Since T is universal in power κ, there exist operations $\cdots F \cdots$, $\cdots G \cdots$, a_i and K_{ij} so that

$\tau_F =$ the given $\cdots F \cdots$,

$\tau_{ij} = F_0$, $\tau'_{ij} = F_1$.

Thus the "found" operations $\cdots F \cdots$ of our solution are none other than the given $\cdots F \cdots$. Moreover, referring to our previous analysis, the fact that $F_0 \neq F_1$ tells us that we must have A_{ij} disjoint from B_{ij} for each i and j. By φ_1, we have $\alpha_i = \beta_i$ for some i. Thus for this i and each j, we can have A_{ij} disjoint from B_{ij} only if $F_3(\gamma_j, \delta_j) \neq F_1$ universally in $x_1, \ldots, x_N$. By our description of F_3 above, we see that this last inequality entails $(\gamma_{j1} = \delta_{j1}) \vee \cdots \vee (\gamma_{jN} = \delta_{jN})$ for each j, universally in $x_1, \ldots, x_N$. Thus we have accomplished our goal of making φ_2 true in $(\kappa; \cdots F \cdots, \cdots G \cdots)$. Q.E.D.

2. Remarks and examples. The first obvious remark allows us to pay no further attention to our requirement that $0, 1 \in X$, since we may obviously find a first order sentence with models in just these two powers.

2.0. The class of all Π^1_2 classes is closed under finite unions and intersections, but not under complementation. For example $\{2^{\aleph_0}\}$ is not a Π^1_2 class, but its complement is. (See Garland **[14]** and Kunen **[22]**. Kunen's result is really a consistency result: it is consistent with ZFC that $\{2^{\aleph_0}\}$ is not Π^1_2, but it might be Π^1_2 anyway, for some accidental reason such as CH, which forces $\{2^{\aleph_0}\} = \{\aleph_1\}$, a Π^1_2 class.)

For our next two remarks we notice that the notion of Π^1_2 class can be enlarged to cover classes defined by infinitary sentences of pure second order logic of the form

$$\psi = \forall \cdots R \cdots \exists \cdots S \cdots \bigwedge_\alpha \varphi_\alpha$$

with possibly infinitely many R's or infinitely many S's or infinitely many α's.

2.1. An obvious extension of the proof in §1 shows that the classes (containing 0, 1) definable in this way *with finitely many R*'s are precisely the classes definable by *universality of infinite sets T of terms*; in fact we will obtain $|T| \leq \aleph_0 + |\psi|$.

2.2. If ψ is *recursively* given, then the Π^1_2 class defined by ψ is equal to the Π^1_2 class defined by some finite sentence, except possibly for finite cardinals. One needs to include a new predicate N, new operations 0, S, + on N, and add the axioms for "Q" relativized to N. Here Q is a finite fragment of Peano's axioms which is strong enough to provide a definition of every recursive function (see e.g. [**46**, p. 51]). One can then code ψ as a finite sentence; for details of this procedure (and some other related procedures) see Kleene [**19**], and Craig and Vaught [**5**].

We can now examine some naturally occurring examples of Π^1_2 classes (sometimes in the form of 2.1 or 2.2 above).

2.3. The class of cardinals κ *not* omitting a type Σ (for some theory Γ). See Morley [**33**] or Proposition 7.2.4 on p. 434 of [**4**]. Morley's results yield the cardinal classes $(\beth_\alpha, \infty)$ for α any countable ordinal; for α recursive we get $(\beth_\alpha, \infty)$ given by a finite ψ, *via* 2.2.

2.4. *Two cardinal problems*. Let Σ be a first order theory in relations $\cdots T \cdots$ with a distinguished unary relation U. We say that Σ *admits* (κ, λ) if there exists a model $(A; U, \ldots)$ of Σ with $|A| = \kappa$ and $|U| = \lambda$. Suppose we are given a second order sentence

$$\psi = \forall \cdots R \cdots \exists \cdots S \cdots \varphi$$

which defines a class X of cardinals. We are interested in the class X' defined by

$$\psi' = \forall U \cdots T \cdots R \cdots \exists \cdots S \cdots \quad (\Sigma \to \varphi^U),$$

with φ^U denoting the relativization of φ to U. It is routine to check that

$$X' = \{\kappa\colon \forall\lambda(\Sigma \text{ admits } (\kappa, \lambda) \to \lambda \in X)\}.$$

See Proposition 3.2.11(iv) on p. 133 of [**4**], and also Exercise 3.2.15, where one will find a finite Σ which admits (κ, λ) iff $\lambda \leq \kappa \leq \aleph_n(\lambda)$. One may easily check that in this case if $X = (\nu, \infty)$ then $X' = (\aleph_n(\nu), \infty)$.

For another example take Σ to say, of $(A; U, T)$, that T maps the n-element subsets of A into U in such a manner that there does not exist any $(n + 1)$-element subset all of whose n-element subsets are mapped to the same place. The "arrow relation" theory of Erdős and Hajnal (Theorem 39(i) of [**12**] and Lemma 5F of [**11**], or Theorem 7.2.1 of [**4**]) says that Σ admits (κ, λ) iff $\lambda \leq \kappa \leq \beth_n(\lambda)$. Taking X as above, we obtain $X' = (\beth_n(\nu), \infty)$.

2.5 *Rigid models of a theory*. Let φ_1 be any first order sentence (or recursive theory) in relations $\cdots R \cdots$ and operations $\cdots F \cdots$, and consider the second order sentence

$$\psi = \forall \cdots F \cdots R \cdots \exists G_1 G_2 (\varphi_1 \to \varphi_2),$$

where φ_2 is the conjunction of the universal closures of these formulas:

$$G_1 G_2 x = G_2 G_1 x = x,$$
$$G_1 F(x_1, x_2, \ldots) = F(G_1 x_1, G_1 x_2, \ldots) \quad (\text{each } F),$$
$$R(x_1, x_2, \ldots) \leftrightarrow R(G_1 x_1, G_1 x_2, \ldots) \quad (\text{each } R),$$
$$\exists x (G_1 x \neq x).$$

Clearly G_1 represents a nonidentity automorphism of a model of φ_1. Thus $\kappa \models \psi$ iff φ_1 has no rigid models of power κ. (Obviously there is a corresponding ψ for endomorphism-rigidity.)

Many examples of rigid spectra have been given by Ehrenfeucht **[6]** and then by Shelah **[40]**. Shelah obtains almost every Π_2^1 class as the complement of a rigid spectrum.

2.6. *Simple algebras, etc.* We may proceed as in 2.5, but take φ_2 to say that certain relations define, e.g., a nontrivial homomorphic image, a nontrivial factorization, an onto endomorphism which is not one-one, and so on. In this way we obtain a Π_2^1 class X whose complement is the class of powers of:

the simple algebras in V (see, e.g., [**30**], [**23**]),
the Hopfian algebras in V (see [**24**], [**25**]),
the directly indecomposable algebras in V, etc.

(V = Mod φ_1). Enlarging $\cdots F \cdots$ to include two new constants F_0, F_1, and taking φ_2 to say that $F_0 = F_1$ or S is a nontrivial congruence separating F_0 and F_1, we obtain the complement of the class of powers of subdirectly irreducible models of φ_1, a class which was studied in **[47]**.

Pseudosimplicity **[32]** may also be handled in like manner.

2.7. *Jónsson algebras.* A *Jónsson algebra* is an infinite algebra with no proper subalgebras of the same power. See pp. 469–470 of Chang and Keisler **[4]**, or pp. 120–135 of Jónsson **[18]** for a survey of this topic, which by and large remains mysterious and caught up with axiomatic questions in set theory. We can summarize, as follows, what is known about the existence of Jónsson algebras.

(1) One unary operation: they exist in power $\aleph_0$ only.

(2) N unary operations ($2 \leqslant N \leqslant \aleph_0$): they exist in powers $\aleph_0$ and $\aleph_1$ only.

(3) λ unary operations ($\lambda \geqslant \aleph_1$): they exist in powers $\aleph_0, \aleph_1, \ldots, \lambda^+$ only.

(4) (Erdős and Hajnal) If there exists a Jónsson algebra of any countable type in power κ, then there also exists a Jónsson algebra (κ; F) with F binary.

(5) (Galvin, Rowbottom, Erdős, Hajnal, Chang) One binary operation: if there exists a Jónsson algebra of power κ, then there exists one of power κ^+; in particular they exist in power $\aleph_0, \aleph_1, \aleph_2, \aleph_3, \ldots$.

(6) (Keisler, Rowbottom) One binary operation: if $V = L$, then they exist in all powers.

(7) (Erdős, Hajnal) One binary operation: if GCH, then for all κ there is a Jónsson algebra of power κ^+. (Refined by Shelah in **[39]**.)

(8) (Erdős, Hajnal) One binary operation: if (κ, F) is a Jónsson algebra, then κ is not measurable.

Clearly the nonexistence of a Jónsson algebra of power κ modeling φ_1 is described by a Π_2^1 class of cardinals, in a manner that should by now be familiar, thereby leading to a finite set *T of terms which is universal in power* $\kappa > 2$ iff *there is no Jónsson algebra of power* κ modeling φ_1. For Mod φ_1 = all algebras (A; F), F binary, this is one of the rare cases where we can write such a T simply and directly. It consists of these eleven terms:

$$0,\quad 1,\quad Fxy,\quad Hxy,\quad Jxyzw,\quad K_1(x, a, Jaacd, J01cd, Haa),\quad K_1(x, \varphi x, c, d, 0),$$
$$K_2(x, 1, Jaacd, J01cd, Haa),\quad K_2(x, H(\psi\varphi x, x), c, d, 0),$$
$$K_3(x, y, 1, Jaacd, J01cd, Haa),\quad K_3(x, y, H(\varphi Gxy, F(\varphi x, \varphi y)), c, d, 0).$$

First, assume these terms are universal in κ; we will show that no $(\kappa;\ F)$ can be a Jónsson algebra. To do this, we will apply the definition to this $F(x, y)$, to 0 and 1 any two distinct elements, to

$$Jxyzw = \begin{cases} z & \text{if } x = y, \\ w & \text{if } x \neq y, \end{cases} \qquad Hxy = \begin{cases} 0 & \text{if } x = y, \\ 1 & \text{if } x \neq y, \end{cases}$$

and with the pair $K_1(\cdots)$, $K_1(\cdots)$ taken to be any two functions which never agree (e.g. the constant functions 0 and 1); likewise for the pairs $(K_2(\cdots),$ $K_2(\cdots))$ and $(K_3(\cdots), K_3(\cdots))$. Take a, c, d, φ, ψ, K_1, K_2, K_3 as supplied by universality. We obviously have

$$(*) \qquad Jaacd = c,\quad J01cd = d,\quad Haa = 0,$$

and so if the K_1-pair, the K_2-pair and the K_3-pair are each to disagree completely, we must have, for all x and y:

$$\varphi x \neq a,\quad H(\psi\varphi x, x) \neq 1,\quad H(\varphi Gxy, F(\varphi x, \varphi y)) \neq 1.$$

Referring to the way we chose H above, we have, for all x and y,

$$(**) \qquad \varphi x \neq a,\quad \psi\varphi x = x,\quad \varphi Gxy = F(\varphi x, \varphi y).$$

These formulas tell us that φ is a one-one mapping onto a proper subalgebra of $(\kappa;\ F)$; thus $(\kappa;\ F)$ is not a Jónsson algebra.

Conversely, let us assume there is no Jónsson algebra of power κ and prove universality on κ. Given eleven operations corresponding to these terms, they first give us F, H, J, 0 and 1. Now if $(*)$ should fail for any a, c, d, then we would be done, since each K_i-pair would be defined on disjoint subsets. Thus we may assume that $(*)$ holds identically, and, in particular, that $0 \neq 1$. Since $(\kappa;\ F)$ is not a Jónsson algebra, there exist a, φ, ψ, G satisfying $(**)$; from $(*)$ we see

$$\varphi x \neq a,\quad H(\psi\varphi x, x) = 0,\quad H(\varphi Gxy, F(\varphi x, \varphi y)) = 0.$$

Again we have each K_i-pair defined on disjoint subsets, and universality is assured. □

Thus this set of eleven terms is universal in a somewhat mysterious set of cardinals, partly described by points 4–8 above. Obviously similar sets can be obtained for the Jónsson algebras in any finitely based variety. Shelah recently found an uncountable Jónsson group (see **[41]**), but almost nothing is known in general (but cf. **[29]**, **[20]** and **[21]**).

2.8. *Examples from set theory, higher cardinals, etc*. The reader is invited to try his hand at representing these as Π^1_2 classes:

strong limit cardinals,
strongly inaccessible cardinals,
Ramsey cardinals,
inaccessible weakly compact cardinals.

The class of measurable cardinals is not so representable, as W. N. Reinhardt has informed us, for it is impossible to define measurability of κ without reference to the power set of κ, or something else at least as large.

All of these may of course be converted to a finite set of terms in the manner of the Main Theorem. But in one case, *regularity*, it is possible to give a nice simple T directly. Namely, κ is regular iff this set of terms is universal in power κ:

$$0,\quad 1,\quad Exy,\quad Sx,\quad Wx,\quad Hxyzw,\quad Jxyzw,$$
$$K(x,\, 1, Jaacd,\, J01cd,\, Haacd,\, Habcc,\, a),$$
$$K(x,\, H(x,\, RSTx,\, x,\, Re(Sa,\, Tx)),\, c,\, d,\, 0,\, 0,\, E(Sa,\, Wa)).$$

The proof is similar to that in 2.7.

2.9. *The finite range.* A good many examples are known concerning universality of T in power $\kappa < \omega$, even for T a single term in unary operations; consult the papers of Isbell, Ehrenfeucht and Silberger. Some other interesting examples can be found earlier in this section, for instance, the set of all integers k which are not the order of any finite simple group (2.6).

First order spectra play an interesting role in the finite range (although, by the Skolem-Löwenheim theorem, they do not discriminate between any infinite cardinals). If φ_1 is a first order sentence in $\cdots F \cdots$ and $\cdots R \cdots$, and φ_2 is a first order sentence in $\cdots G \cdots$ and $\cdots S \cdots$, then $(\forall \cdots F \cdots R \cdots)(\exists \cdots G \cdots S \cdots)(\varphi_1 \to \varphi_2)$ holds precisely on Spec φ_2 $\cup$ (Complement Spec φ_1). For $\varphi_1 = \top$, we get Spec φ_2, and for $\varphi_2 = \bot$, we get (Complement Spec φ_1) $\cup$ $\{0, 1\}$.

Again we may bypass the Main Theorem in one special case, this time representing $\{0, 1\} \cup$ (Complement Spec φ_1) as $\{\kappa$: T is universal in power $\kappa\}$ for a very easily described T. We will assume that φ_1 is the conjunction of identities $\sigma_i = \tau_i$ $(1 \leqslant i \leqslant N)$. The following terms are obviously universal in power κ iff $\kappa = 0$ or 1 or $\kappa \notin$ Spec φ_1:

$$T\begin{cases} F(x_1, x_2, \ldots) \quad (\text{any } F \text{ of } \varphi_1), \\ K(\sigma_1(a_1, a_2, \ldots), \ldots, \sigma_N(a_1, a_2, \ldots)), \\ K(\tau_1(a_1, a_2, \ldots), \ldots, \tau_N(a_1, a_2, \ldots)). \end{cases}$$

By the way, this construction of T makes the Corollary obvious. For T is obviously recursive in φ_1, and it is undecidable whether $\omega \in$ Spec φ_1 (by Perkins [35]).

2.10. *Unions, revisited.* If T and T' are finite sets of terms, then the Main Theorem (together with 2.0) tells us that $\{\kappa$: T universal in $\kappa\} \cup \{\kappa$: T' universal in $\kappa\}$ must be of the form $\{\kappa$: T'' universal in power $\kappa\}$ for some finite set T'', but it is not immediately apparent how to get a simple description of T''. In the special case $T = \{\tau(x, y)\}$ and $T' = \{\sigma(u, v)\}$ (with no function symbols appearing in both), the reader may enjoy checking that the following will do for T'':

$$0,\quad 1,\quad Jxyuv,\quad Hxyuv,\quad Fxy,\quad Guv,$$
$$K(x, y, u, v,\, 1,\, Haacd,\, Habcc,\, Jaacd,\, J01cd),$$
$$K(x, y, u, v,\, H(Fxy,\, \tau(x, y),\, Guv,\, \sigma(u, v)),\, 0,\, 0,\, c,\, d).$$

Problem. Is there a map $(T, T') \mapsto T''$ such that, *for all clones* C, T'' is universal on C iff T or T' is universal on C?

Our T'' works for the clones $C(\kappa)$, but seems to use special properties of those clones. Of course, if "or" is replaced by "and", the answer is obviously yes. (T'' is a "disjoint union" of T and T'.)

2.11. It is fairly easy to get $[0, \aleph_0] \cup [2^{\aleph_0}, \infty)$ as a Π^1_2 class. Thus our Theorem gives us a finite set of terms which is universal in all powers iff CH is true.

3. A special example, not in the style of §1. First, we should point out that Isbell discovered [**15**] that the single term f^2g^2fx is universal in all finite powers but not in any infinite power.

Now we give an example which yields $\{0, 1\} \cup [\aleph_0, \infty)$ in a way that seems different from the examples that have come before. We claim that the set f_1Fxy, f_2Fxy is universal in power κ iff $\kappa = 0, 1$ or $\kappa \geqslant \aleph_0$. Certainly, if this set is universal, then there exist F, f_1, f_2 obeying $f_1Fxy = x$, $f_2Fxy = y$, which implies that $F\colon A^2 \to A$ is one-one, and thus $|A| \leqslant 1$ or $|A| \geqslant \aleph_0$. On the other hand, assume A is infinite and let $F\colon A^2 \to A$ be any one-one map. It is then easy to see how to solve $f_1Fxy = F_1xy$, $f_2Fxy = F_2xy$ for f_1 and f_2.

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